High Altitude Observatory — MEMORANDUM

54 Years • 1940 – 1994

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SUBJECT: A Description of Polarization Modulation and Demodulation

DATE: October 24, 1994

In polarimetry the objective is to measure each element of a Stokes vector entering the polarimeter. There are four elements of the vector, I, Q, U, and V. I is the total intensity. Q is the difference in intensities measured through a perfect linear polarizer oriented at 0° and 90°. U is the difference in the intensities measured through a perfect polarizer oriented at 45° and ~~90°~~ 135°. V is the difference in intensities measured through a quarter wave retarder with a fast axis at 0° followed by a linear polarizer oriented at 45° and 135°. Angles are measured counter clockwise looking at the source. A single linear polarizer can be called a linear analyzer. A quarter wave retarder followed by a linear polarizer at 45° is a left circular analyzer. When the linear polarizer is at 135° it is a right circular analyzer.

Photon detectors such as charge coupled devices (CCDs) are sensitive to intensity only. Such detectors can be used with linear and circular analyzers as described above to measure the values of the Stokes parameters of a beam of light.

There are other measurement techniques which employ different combinations of retarders and linear polarizers. In all cases presented the last element is a linear analyzer. The preceding elements are adjusted by changing optical elements, orientations, or magnitude of retardance, and collectively are called the modulator. Each different optical configuration used to measure polarization is called a modulation state. Since there are four unknowns, the Stokes parameters, at least four measurements of different modulation states are required.

Some types of polarization modulators/analyzers:

1. Stokes definition device: Linear analyzer at four orientations, a left circular analyzer, and a right circular analyzer.

2. Rotational modulator: A retarder or retarders at various rotational angles followed by a linear polarizer.

3. Variable retarder modulator: Two retarders with adjustable retardance with orientations of 0° and 45°, followed by a linear polarizer.

Each of the types shown here use a linear polarizer as the analyzer. If a polarizing beam splitter is used instead, two beams are created, each of which can be used to measure polarization. The two beams may be used together by a single instrument to increase signal to noise (Lites, 1987) or may be used to feed two different instruments.

The measured Stokes vector is the product of matrices which describe the polarization properties of the optical elements acting on the input Stokes vector. These matrices are called Mueller matrices. A matrix equation is created for each of the modulation states.

measured — input

Only one we measure

$$\begin{pmatrix} i_j \\ q_j \\ u_j \\ v_j \end{pmatrix} = A_j \ M_j \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

where A is the analyzer Mueller matrix, M is the modulator Mueller matrix, i_j, q_j, u_j and v_j are elements of the measured Stokes vector, and j indicates the particular modulation state.

Sums and differences of the measurements at the different modulation states are used to "demodulate" the measurements to create the measured Stokes vector. Following are examples of how this is done.

Stokes Parameter Modulator/Analyzer:

This is the modulator/analyzer described earlier which is used in the definition of the Stokes parameters. There are six modulation states. The first four use a linear polarizer oriented at 0°, 45°, 90°, and 135°. State five is a quarter wave retarder with fast axis at 0° followed by a linear polarizer at 45°. State six is a quarter wave retarder with fast axis at 0° followed by a linear polarizer at 135°.

The first four states are described using the Mueller matrix for a linear polarizer. The general form of the matrix is shown with the orientation at an angle, θ.

(L)

$$\begin{pmatrix} i_j \\ q_j \\ u_j \\ v_j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & c_{2\theta} & s_{2\theta} & 0 \\ c_{2\theta} & \frac{1}{2}(1 + c_{4\theta}) & \frac{1}{2}s_{4\theta} & 0 \\ s_{2\theta} & \frac{1}{2}s_{4\theta} & \frac{1}{2}(1 - c_{4\theta}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

agrees with SPIE formula

where θ is the orientation of the linear polarizer and $c_x \equiv \cos(x)$ and $s_x \equiv \sin(x)$.

$$i_j = \frac{1}{2}(I + c_{2\theta}Q + s_{2\theta}U)$$

The four states are $\theta = 0°, 45°, 90°$ and $135°$.

Q=

$i_1 = \frac{1}{2}\,(I+Q)$ 0

$i_2 = \frac{1}{2}\,(I-Q)$ 90

$i_3 = \frac{1}{2}\,(I+U)$ 45

$i_4 = \frac{1}{2}\,(I-U)$ 135

States five and six can be derived using the matrix form for a quarter wave retarder at 0° and for a linear polarizer at 45° and 135°.

$$s = L(\theta_j)D(\varphi_j)S$$

where L is as defined above and

$$D(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a + bc_{4\varphi} & bS_{4\varphi} & -s_{2\varphi}s\delta \\ 0 & bs_{4\varphi} & a - bc_{4\varphi} & c_{2\varphi}s\delta \\ 0 & s_{2\varphi}s\delta & -c_{2\varphi}s\delta & c\delta \end{pmatrix}$$

where the linear polarizer matrix is as above, δ is the retardance of the retarder (45°), φ is the orientation of the retarder (0°), $a \equiv \frac{1+c\delta}{2}$ and $b \equiv \frac{1-c\delta}{2}$. The orientations of the linear polarizer are $\theta = 45°$ and 135°. Using those angles and performing the matrix algebra states 5 and 6 are derived.

$i_5 = \frac{1}{2}\,(I+V)$

$i_6 = \frac{1}{2}\,(I-V)$

Demodulation is achieved by summing the modulation states in the following manner.

i	+	+	+	+	+	+
q	+	−	0	0	0	0
u	0	0	+	−	0	0
v	0	0	0	0	+	−

$$\begin{aligned} i &= i_1 + i_2 + i_3 + i_4 + i_5 + i_6 &&= 3I \\ q &= i_1 - i_2 &&= Q \\ u &= i_3 - i_4 &&= U \\ v &= i_5 - i_6 &&= V \end{aligned}$$

It is no surprise that the measured q, u, and v are just the input Q, U, and V, since this polarimeter follows the definitions. The modulation efficiency is computed by dividing the measured Stokes elements by the sum over all the modulation states of the input Stokes elements. Since there are six states the sum of the input elements are $6I$, $6Q$, $6U$, and $6V$. The modulation efficiencies are

$$\begin{aligned} \text{eff}_I &= \frac{3I}{6I} &= 0.5 \\ \text{eff}_Q &= \frac{Q}{6Q} &= 0.167 \\ \text{eff}_U &= \frac{U}{6U} &= 0.167 \\ \text{eff}_V &= \frac{V}{6V} &= 0.167 \end{aligned}$$

A polarimeter can be expressed as polarization response matrix X acting upon the input Stokes vector S producing the measured Stokes vector s.

$$s = XS$$

In the case of a Stokes definition modulator analyzer:

$$X = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & .333 & 0 & 0 \\ 0 & 0 & .333 & 0 \\ 0 & 0 & 0 & .333 \end{pmatrix}$$

Note the diagonal terms show the modulation efficiency. The input Stokes vector is recovered from the measured vector using the inverse polarimeter response matrix.

$$S = X^{-1}s$$

Retarder at Various Angles Followed by a Linear Polarizer:

This polarimeter uses a retarder rotated to various specific angles followed by a linear polarizer at 0°. Note that in this and following cases the orientation of the analyzer if fixed. The measured Stokes elements are computed using the Mueller matrix forms for a retarder and linear polarizer acting upon the input Stokes vector.

For the first couple of examples with this modulator, a retardance of a quarter wave (90°) will be used.

$$\begin{aligned} s &= L\ (0^\circ)\ D\ (\varphi, \lambda/4)\ S \\ i_j &= \tfrac{1}{2}\ [I + (\tfrac{1}{2} + \tfrac{1}{2}C_{4\varphi})\ Q + \tfrac{1}{2}S_{4\varphi}\ U - S_{2\varphi}\ V] \end{aligned}$$

With this polarimeter, eight modulation states are used. If these are at modulator positions of $11.25° + \jmath \times 22.5°$, then the following measurements are made:

$$
\begin{aligned}
i_1 &= \tfrac{1}{2}\,[I + .853Q + .354U - .354V] \\
i_2 &= \tfrac{1}{2}\,[I + .145Q + .354U - .924V] \\
i_3 &= \tfrac{1}{2}\,[I + .145Q - .354U - .924V] \\
i_4 &= \tfrac{1}{2}\,[I + .853Q - .354U - .354V] \\
i_5 &= \tfrac{1}{2}\,[I + .853Q + .354U + .354V] \\
i_6 &= \tfrac{1}{2}\,[I + .145Q + .354U + .924V] \\
i_7 &= \tfrac{1}{2}\,[I + .145Q - .354U + .924V] \\
i_8 &= \tfrac{1}{2}\,[I + .853Q - .354U + .354V]
\end{aligned}
$$

Demodulation is performed by summing states with the following signs

i:	+	+	+	+	+	+	+	+
q:	+	−	−	+	+	−	−	+
u:	+	+	−	−	+	+	−	−
v:	−	−	−	−	+	+	+	+

Since there are eight states, one must divide by eight times the input Stokes vector to compute the polarimeter response matrix

$$
X = \frac{1}{2}\begin{pmatrix} 1 & 0.5 & 0 & 0 \\ 0 & .354 & 0 & 0 \\ 0 & 0 & .354 & 0 \\ 0 & 0 & 0 & .654 \end{pmatrix}
$$

Consider a modulator with retardance other than 90°. A particularly useful retardance value for a retarder which is stepped to specific angles is 123.15°. Following the same analysis as was applied to the quarter wave case, the polarimeter response matrix is computed.

$$
X = \frac{1}{2}\begin{pmatrix} 1.0 & .266 & 0 & 0 \\ 0 & .546 & 0 & 0 \\ 0 & 0 & .546 & 0 \\ 0 & 0 & 0 & .546 \end{pmatrix}
$$

Modulation efficiencies are much more closely matched for this technique.

Rather than stepping the retarder to various positions, what if the retarder is continuously rotated during course of the measurement? In this case the samples are integrals over intervals. The ones chosen start at $0° + \jmath \times 22.5°$ and are 22.5° wide.

$$i = \tfrac{1}{2}\left\{ I \int_{\varphi_1}^{\varphi_2} d\varphi + Q \int_{\varphi_1}^{\varphi_2} \Big(a + b\cos(4\varphi)\Big) d\varphi + U \int_{\varphi_1}^{\varphi_2} b\sin(4\varphi) d\varphi - V \int_{\varphi_1}^{\varphi_2} \sin(2\varphi)\sin(\delta) d\varphi \right\}$$

$$i = \tfrac{1}{2}\left\{ I\Big[\varphi\Big]_{\varphi_1}^{\varphi_2} + aQ\Big[\varphi\Big]_{\varphi_1}^{\varphi_2} + \tfrac{b}{4}Q\Big[\sin(4\varphi)\Big]_{\varphi_1}^{\varphi_2} - \tfrac{b}{4}U\Big[\cos(4\varphi)\Big]_{\varphi_1}^{\varphi_2} + V\tfrac{1}{2}\sin(\delta)\Big[\cos(2\varphi)\Big]_{\varphi_1}^{\varphi_2} \right\}$$

$$X = \tfrac{1}{2}\begin{pmatrix} 1 & .124 & 0 & 0 \\ 0 & .318 & 0 & 0 \\ 0 & 0 & .318 & 0 \\ 0 & 0 & 0 & .636 \end{pmatrix}$$

The modulation efficiency is slightly reduced compared to a stepped retarder. This type of modulator is commonly used since a continuous rotation is easier to perform mechanically, especially when high speed is required.

For a continuously rotating retarder there is again an angle which matches the efficiencies of the Stokes elements. That angle is 126.85° and produces an efficiency of .255.

$$X = \tfrac{1}{2}\begin{pmatrix} 1 & .050 & 0 & 0 \\ 0 & .510 & 0 & 0 \\ 0 & 0 & .510 & 0 \\ 0 & 0 & 0 & .510 \end{pmatrix}$$

It is possible to build a polarimeter using a couple of quarter wave retarders, each of which is rotated to different angles to create the modulation states.

Two Variable Retarders Followed by a Linear Polarizer:

This modulator/analyzer uses a variable retarder γ with fast axis at 0°, a variable retarder δ with fast axis at 45° and a linear polarizer at 0°. The equation for the output intensity is

$$i = \tfrac{1}{2}\left[I + C_\delta\, Q + S_\delta S_\gamma\, U - S_\delta C_\gamma\, V\right]$$

Many thanks to Jorge Sanchez Almeida of the Institutio de Astrofisica de Canarias for the following angles:

State	γ	δ
1	135.00°	125.27°
2	-45.00°	125.27°
3	45.00°	54.73°
4	45.00°	-54.73°

The resulting measurements are

$$X = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & .577 & 0 & 0 \\ 0 & 0 & .577 & 0 \\ 0 & 0 & 0 & .577 \end{pmatrix}.$$

Other Issues:

The above discussion assumes perfect optical elements used for the modulator and analyzer. This assumption is adequate to understand of how a polarimeter works, but once a system is selected, an analysis using measured or predicted actual values for the actual Mueller matrices should be performed. In addition, any polarization contributions from other optical elements, not intended to be involved in the polarization measurement, must be included.

A full description of the polarimeter must also include the gain and offset of the detector as well as the detection scheme (Seagraves and Elmore, 1994) and can be performed on possible optical designs.

BIBLIOGRAPHY

[1] B. W. Lites, "Rotating waveplates as polarization modulators for Stokes polarimetry of the Sun: Evaluation of seeing-induced crosstalk errors," *Applied Optics*, **26**, pp. 3838-3845, 1987.

[2] P. H. Seagraves and D. F. Elmore, "Use of Mueller and non-Mueller matrices to describe polarization properties of telescope-based polarimeters," *SPIE Proceedings: Polarizastion Analysis and Measurement II*, **2265**, pp. 231-238, 1994.

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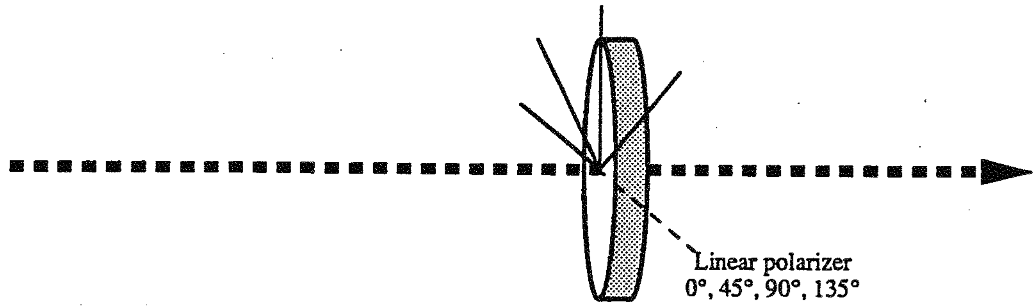
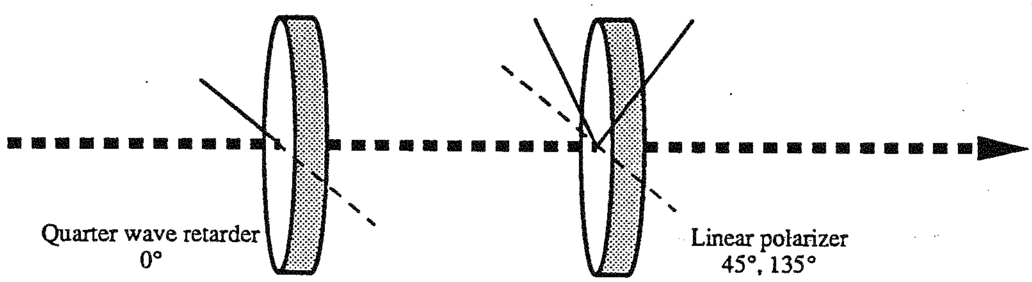
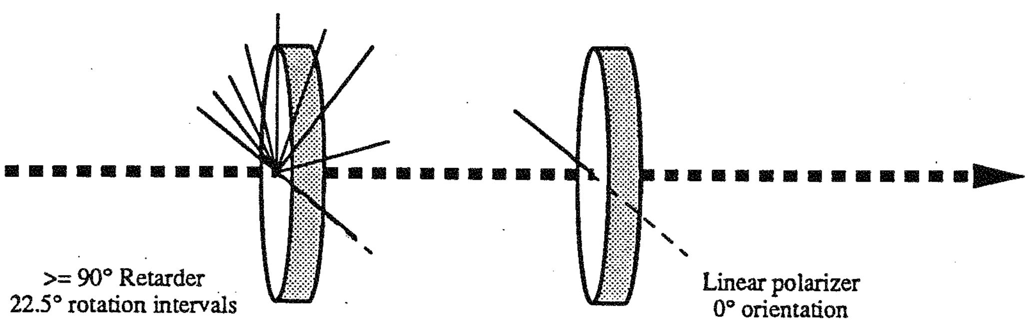
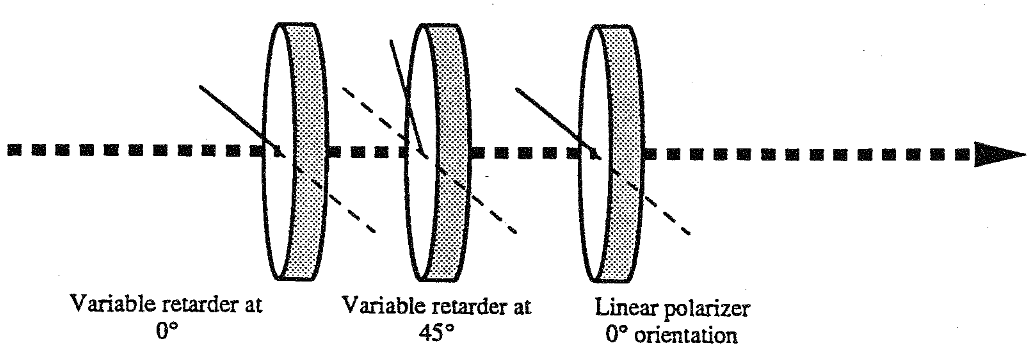
Some Polarimeter Configurations

18 Oct 1994









Stepped Waveplate Polarimeter

i = 1/2 [I + (a+bcos(4phi))Q + bsin(4phi) U - s(2phi) s(delta) V]

a = (1 + cos(delta))/2
b = (1- cos(delta))/2

					a	b
delta	**90.000**	degrees	1.571	radians	0.500	0.500
phi offset	**11.250**	degrees	0.196	radians		

	Phi		i	q	u	v
1	11.250	0.19634954	0.500	0.427	0.177	-0.191
2	33.750	0.58904861	0.500	0.073	0.177	-0.462
3	56.250	0.98174769	0.500	0.073	-0.177	-0.462
4	78.750	1.37444676	0.500	0.427	-0.177	-0.191
5	101.250	1.76714584	0.500	0.427	0.177	0.191
6	123.750	2.15984491	0.500	0.073	0.177	0.462
7	146.250	2.55254399	0.500	0.073	-0.177	0.462
8	168.750	2.94524306	0.500	0.427	-0.177	0.191

		0.500	0.250	0.000	0.000
Polarimeter response matrix	X =	0.000	0.177	0.000	0.000
		0.000	0.000	0.177	0.000
		0.000	0.000	0.000	0.327

Stepped Waveplate Polarimeter

i = 1/2 [I + (a+bcos(4phi))Q + bsin(4phi) U - s(2phi) s(delta) V]

a = (1 + cos(delta))/2
b = (1- cos(delta))/2

					a	b
delta	**123.150**	degrees	2.149	radians	0.227	0.773
phi offset	**11.250**	degrees	0.196	radians		

	Phi		i	q	u	v
1	11.250	0.19634954	0.500	0.387	0.273	-0.160
2	33.750	0.58904861	0.500	-0.160	0.273	-0.387
3	56.250	0.98174769	0.500	-0.160	-0.273	-0.387
4	78.750	1.37444676	0.500	0.387	-0.273	-0.160
5	101.250	1.76714584	0.500	0.387	0.273	0.160
6	123.750	2.15984491	0.500	-0.160	0.273	0.387
7	146.250	2.55254399	0.500	-0.160	-0.273	0.387
8	168.750	2.94524306	0.500	0.387	-0.273	0.160

		0.500	0.113	0.000	0.000
Polarimeter response matrix	X =	0.000	0.273	0.000	0.000
		0.000	0.000	0.273	0.000
		0.000	0.000	0.000	0.273

Rotating Waveplate Polarimeter

i = 1/2{I[phi2-phi1] + Q[a(phi2-phi1) + b(sin(4phi2) -sin(4phi1))/4] + U[b(cos(4phi2)-cos(4phi1))/4] + V[sin(delta)(cos(2phi2)-cos(2phi1))]}

a = (1 + cos(delta))/2
b = (1- cos(delta))/2

					a	b
delta	**126.850**	degrees	2.214	radians	0.200	0.800
phi offset	**0.000**	degrees	0.000	radians		

State	Phi	Phi (radians)	i	q	u	v
1	0.000	0	0.196	0.139	0.100	-0.059
2	22.500	0.39269908	0.196	-0.061	0.100	-0.141
3	45.000	0.78539815	0.196	-0.061	-0.100	-0.141
4	67.500	1.17809723	0.196	0.139	-0.100	-0.059
5	90.000	1.5707963	0.196	0.139	0.100	0.059
6	112.500	1.96349538	0.196	-0.061	0.100	0.141
7	135.000	2.35619445	0.196	-0.061	-0.100	0.141
8	157.500	2.74889353	0.196	0.139	-0.100	0.059
9	180.000	3.1415926				

Demodulation	I	++++++++
	Q	+--++--+
	U	++--++--
	V	----++++

		0.500	0.100	0.000	0.000
		0.000	0.255	0.000	0.000
Polarimeter response matrix	X =	0.000	0.000	0.255	0.000
		0.000	0.000	0.000	0.255

Rotating Waveplate Polarimeter

i = 1/2 [I(phi2-phi1) + a(phi2-phi1) + b(cos(4phi2)-cos(4phi1)) + b(sin(4phi2) -sin(4phi1)) + sin(delta)(cos(2phi2) - cos(2phi1))]

a = (1 + cos(delta))/2
b = (1- cos(delta))/2

					a	b
delta	**90.000**	degrees	1.571	radians	0.500	0.500
phi offset	**0.000**	degrees	0.000	radians		

State	Phi	Phi (radians)	i	q	u	v
1	0.000	0	0.196	0.087	0.062	-0.073
2	22.500	0.39269908	0.196	-0.038	0.062	-0.177
3	45.000	0.78539815	0.196	-0.038	-0.062	-0.177
4	67.500	1.17809723	0.196	0.087	-0.063	-0.073
5	90.000	1.5707963	0.196	0.087	0.062	0.073
6	112.500	1.96349538	0.196	-0.038	0.063	0.177
7	135.000	2.35619445	0.196	-0.038	-0.062	0.177
8	157.500	2.74889353	0.196	0.087	-0.063	0.073
9	180.000	3.1415926				

		0.500	0.062	0.000	0.000
		0.000	0.159	0.000	0.000
Polarimeter response matrix	X =	0.000	0.000	0.159	0.000
		0.000	0.000	0.000	0.318

Two Stepped Waveplates + Linear Polarizer Polarimeter

First wave plate at alpha degrees, second at beta degrees, both quarter wave retarndance

i = 1/2 [I + cos(2beta)(Qcos(2alpha) + Usin(2alpha)) (from ISAS meeting)
-Qcos(2(beta-alpha)) + Usin(2(beta-alpha))
+cos(2beta)cos(2(beta-alpha))(Qcos(2alpha) + Usin(2alpha))
+ V(sin(2beta)cos(2(beta-alpha)) - sin(2alpha))]

delta **90.000** degrees 1.571 radians

	alpha	alpha (radians)	beta	beta (radians)	i	q	u	v
1	135.000	2.35619445	45	0.78539815	0.500	0.500	0.000	0.000
2	45.000	0.78539815	45	0.78539815	0.500	-0.500	0.000	0.000
3	0.000	0	45	0.78539815	0.500	0.000	0.500	0.000
4	90.000	1.5707963	45	0.78539815	0.500	0.000	-0.500	0.000
5	135.000	2.35619445	0	0	0.500	-0.354	0.000	-0.500
6	45.000	0.78539815	90	1.5707963	0.500	-0.354	0.000	0.500

Polarimeter response matrix X =

++++++	0.500	-0.118	0.000	0.000
+-0000	0.000	0.167	0.000	0.000
00+-00	0.000	0.000	0.167	0.000
0000-+	0.000	0.000	0.000	0.167